Probabilistic models
-
Bayesian Regression

Marylou Gabrié
Slides based on Lecture 08b from David Rosenberg's course materials
(https://github.com/davidrosenberg/mlcourse)

CDS, NYU

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1. Recap: Conditional Probability Models
2. Bayesian Conditional Probability Models
3. Gaussian Regression Example
4. Gaussian Regression Continued

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Conditional Probability Modeling

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- **Outcome space** $\mathcal{Y}$
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- A **parametric family of conditional densities** is a set

$$\{p(y \mid x, \theta) : \theta \in \Theta\},$$

  - where $p(y \mid x, \theta)$ is a density on **outcome space** $\mathcal{Y}$ for each x in **input space** $\mathcal{X}$, and
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  - θ is a **parameter** in a [finite dimensional] **parameter space** Θ**.**

- This is the common starting point for a treatment of classical or Bayesian statistics.

Likelihood Function

- **Data:** $\mathcal{D} = (y_1, \ldots, y_n)$
- The probability density for our data $\mathcal{D}$ is

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- For fixed $\mathcal{D}$, the function $\theta \mapsto p(\mathcal{D} \mid x, \theta)$ is the **likelihood function**:

$$L_{\mathcal{D}}(\theta) = p(\mathcal{D} \mid x, \theta),$$

where $x = (x_1, \ldots, x_n)$.

Maximum Likelihood Estimator

- The **maximum likelihood estimator (MLE)** for θ in the family $\{p(y \mid x, \theta) \mid \theta \in \Theta\}$ is

$$\hat{\theta}_{\text{MLE}} = \underset{\theta \in \Theta}{\arg\max}\, L_{\mathcal{D}}(\theta).$$

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- We can think of this as a choice of a particular function from the hypothesis space

$$\mathcal{F} = \{p(y \mid x, \theta) : \theta \in \Theta\}.$$

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  - A **parametric family of conditional densities**:

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  - A **prior distribution** $p(\theta)$ on $\theta \in \Theta$.

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- Posterior represents the **rationally "updated" beliefs** after seeing $\mathcal{D}$.
- Each θ corresponds to a prediction function,
  - i.e. the conditional distribution function $p(y \mid x, \theta)$.

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- As discussed in last video, we may want to use
  - $\hat{\theta} = \mathbb{E}[\theta \mid \mathcal{D}, x]$ (the posterior mean estimate)
  - $\hat{\theta} = \text{median}[\theta \mid \mathcal{D}, x]$
  - $\hat{\theta} = \arg\max_{\theta \in \Theta} p(\theta \mid \mathcal{D}, x)$ (the MAP estimate)
- depending on our loss function.

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- Having set our Bayesian model, how do we predict a distribution on y for input x?
- **There is no selection** from hypothesis space.

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- Such an average is also called a **mixture distribution**.

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Comparison to Frequentist Approach

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- In frequentist approach, we choose $\hat{\theta} \in \Theta$, and predict

$$p(y \mid x, \hat{\theta}(\mathcal{D})).$$

- In Bayesian approach, we integrate out over Θ w.r.t. $p(\theta \mid \mathcal{D})$ and predict with

$$p(y \mid x, \mathcal{D}) = \int p(y \mid x; \theta) p(\theta \mid \mathcal{D})\, d\theta$$

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- Each of these can be derived from $p(y \mid x, \mathcal{D})$.

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Example in 1-Dimension: Setup

- Input space $\mathcal{X} = [-1, 1]$ Output space $\mathcal{Y} = \mathbf{R}$
- Given x, the world generates y as

$$y = w_0 + w_1 x + \varepsilon,$$

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- Written another way, the **conditional probability model** is

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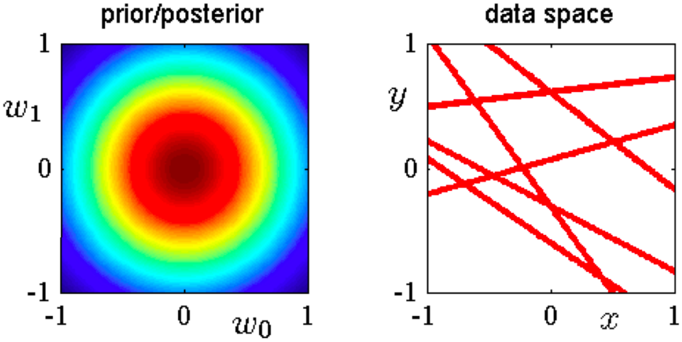
- What's the parameter space? $\mathbf{R}^2$.
- **Prior distribution:** $w = (w_0, w_1) \sim \mathcal{N}\left(0, \frac{1}{2} I\right)$

Example in 1-Dimension: Prior Situation

- **Prior distribution:** $w = (w_0, w_1) \sim \mathcal{N}\left(0, \frac{1}{2}I\right)$ (Illustrated on left)



Bishop's PRML Fig 3.7

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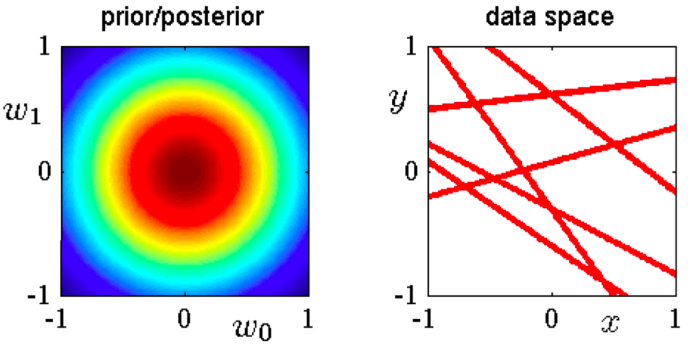
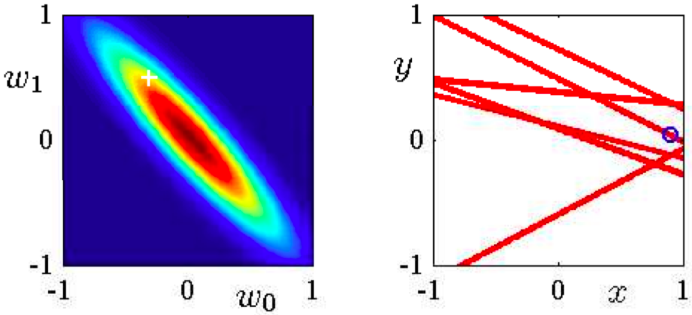


- On right, $y(x) = \mathbb{E}[y \mid x, w] = w_0 + w_1 x$, for randomly chosen $w \sim p(w) = \mathcal{N}\left(0, \frac{1}{2}I\right)$.

Bishop's PRML Fig 3.7

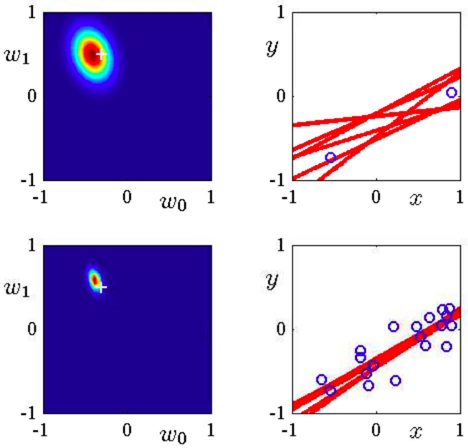
Example in 1-Dimension: 1 Observation



- On left: posterior distribution; white '+' indicates true parameters
- On right:
  - blue circle indicates the training observation
  - red lines, $y(x) = \mathbb{E}[y \mid x, w] = w_0 + w_1 x$, for randomly chosen $w \sim p(w|\mathcal{D})$ (posterior)

Bishop's PRML Fig 3.7

Example in 1-Dimension: 2 and 20 Observations



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- **Posterior Variance** Σ_P gives us a natural **uncertainty measure.**

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which is of course the ridge regression solution.

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- Which is the ridge regression objective.

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- For Gaussian regression, predictive distribution has closed form.

Closed Form for Predictive Distribution

- **Model**:

$$w \sim \mathcal{N}(0, \Sigma_0)$$

Closed Form for Predictive Distribution

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$$\begin{array}{rcl} w & \sim & \mathcal{N}(0, \Sigma_0) \\ y_i \mid x, w & \text{i.i.d.} & \mathcal{N}(w^T x_i, \sigma^2) \end{array}$$

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$$y_{\text{new}} \mid x_{\text{new}}, \mathcal{D} \sim \mathcal{N}\left(\eta_{\text{new}}, \sigma^2_{\text{new}}\right)$$

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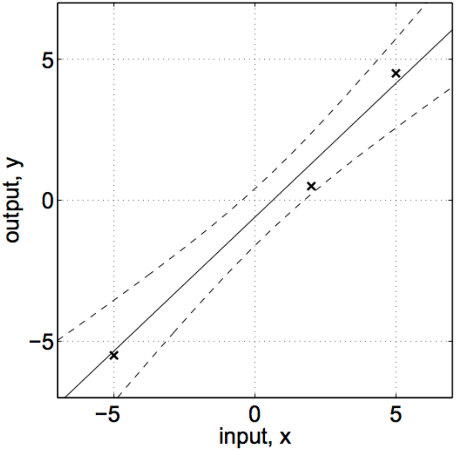
\sigma^2_{\text{new}} & = & \underbrace{x_{\text{new}}^T \Sigma_P x_{\text{new}}}_{\text{from variance in } w} + \underbrace{\sigma^2}_{\text{inherent variance in } y}
\end{aligned}
$$

Predictive Distributions

- With predictive distributions, can give mean prediction with error bands:



Rasmussen and Williams' *Gaussian Processes for Machine Learning*, Fig.2.1(b)