# Probabilistic models
-
# Bayesian Methods

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Slides based on Lecture 08a from David Rosenberg's course materials
(https://github.com/davidrosenberg/mlcourse)

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# Contents

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# Parametric Family of Densities

- A **parametric family of densities** is a set

$$\{p(y \mid \theta) : \theta \in \Theta\},$$

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- This is the common starting point for a treatment of classical or Bayesian statistics.

# Density vs Mass Functions

- In this lecture, whenever we say "density", we could replace it with "mass function."
- Corresponding integrals would be replaced by summations.
- (In more advanced, measure-theoretic treatments, they are each considered densities w.r.t. different base measures.)

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- Instead of $\theta$, we have data $\mathcal{D}$: $y_1, \ldots, y_n$ sampled i.i.d. $p(y \mid \theta)$.
- Statistics is about how to get by with $\mathcal{D}$ in place of $\theta$.

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- **Maximum likelihood estimators** are consistent and efficient under reasonable conditions.

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- Note that every $\theta \in \Theta$ gives us a different probability model for a coin.

# Coin Flipping: Likelihood function

- Data $\mathcal{D} = (H, H, T, T, T, T, T, H, \ldots, T)$, assumed i.i.d. flips.
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$$L_{\mathcal{D}}(\theta) = p(\mathcal{D} \mid \theta) = \theta^{n_h} (1-\theta)^{n_t}$$

- As usual, easier to maximize the log-likelihood function:

$$\begin{aligned} \hat{\theta}_{\text{MLE}} &= \underset{\theta \in \Theta}{\arg\max} \log L_{\mathcal{D}}(\theta) \\ &= \underset{\theta \in \Theta}{\arg\max} [n_h \log \theta + n_t \log(1-\theta)] \end{aligned}$$

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$\hat{\theta}_{\text{MLE}}$ is the empirical fraction of heads.

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- A prior reflects our belief about $\theta$, **before seeing any data**..

# A Bayesian Model

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  1. A parametric family of densities
  $$\{p(\mathcal{D} \mid \theta) \mid \theta \in \Theta\}.$$
  2. A **prior distribution** $p(\theta)$ on parameter space $\Theta$.
- Putting pieces together, we get a joint density on $\theta$ and $\mathcal{D}$:

$$p(\mathcal{D}, \theta) = p(\mathcal{D} \mid \theta)p(\theta).$$

# The Posterior Distribution

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- Prior represents belief about $\theta$ before observing data $\mathcal{D}$.
- Posterior represents the **rationally "updated" belief** about $\theta$, after seeing $\mathcal{D}$.

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- Where $\propto$ means we've dropped factors independent of $\theta$.

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- Need a prior distribution $p(\theta)$ on $\Theta = (0, 1)$.
- A distribution from the Beta family will do the trick...

# Coin Flipping: Beta Prior

- **Prior:**

$$\begin{aligned} \theta &\sim \text{Beta}(\alpha, \beta) \\ p(\theta) &\propto \theta^{\alpha-1}(1-\theta)^{\beta-1} \end{aligned}$$

Figure by Horas based on the work of Krishnavedala (Own work) [Public domain], via Wikimedia Commons http://commons.wikimedia.org/wiki/File:Beta_distribution_pdf.svg.

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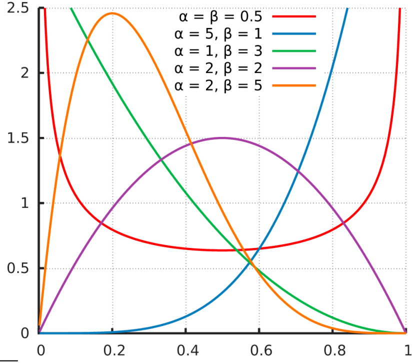


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- **Mode of Beta distribution:**

$$\underset{\theta}{\arg\max}\, p(\theta) = \frac{h-1}{h+t-2}$$

for $h,t>1$.

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- **Interpretation**:
  - Prior initializes our counts with $h$ heads and $t$ tails.
  - Posterior increments counts by observed $n_h$ and $n_t$.

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- The family of all probability distributions is conjugate to any parametric model. [Trvially]

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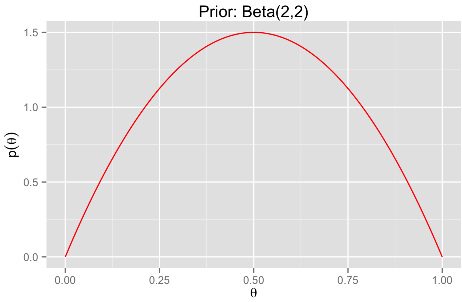

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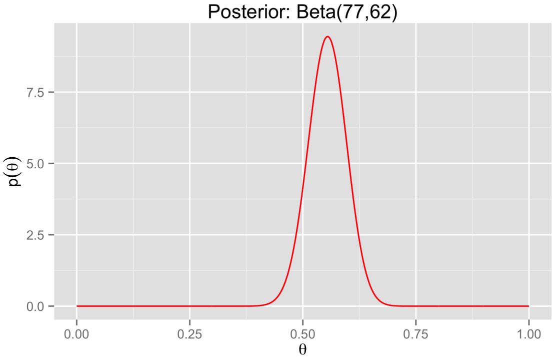
- Heads: 75 Tails: 60
  - $\hat{\theta}_{\text{MLE}} = \frac{75}{75+60} \approx 0.556$
- **Posterior distribution:** $\theta \mid \mathcal{D} \sim \textbf{Beta}(77, 62)$**:**



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- Common options:
    - **posterior mean** $\hat{\theta} = \mathbb{E}[\theta \mid \mathcal{D}]$
    - **maximum a posteriori (MAP) estimate** $\hat{\theta} = \arg\max_\theta p(\theta \mid \mathcal{D})$
        - Note: this is the **mode** of the posterior distribution

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- The most "Bayesian" approach is **Bayesian decision theory**:
  - Choose a loss function.
  - Find action **minimizing expected risk w.r.t. posterior**

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# Bayesian Decision Theory

- Ingredients:
  - **Parameter space** $\Theta$.
  - **Prior**: Distribution $p(\theta)$ on $\Theta$.
  - **Action space** $\mathcal{A}$.
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  - It's the **expected loss under the posterior.**

# Bayesian Decision Theory

- Ingredients:
  - **Parameter space** $\Theta$.
  - **Prior**: Distribution $p(\theta)$ on $\Theta$.
  - **Action space** $\mathcal{A}$.
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  - It's the **expected loss under the posterior.**
- A **Bayes action** $a^*$ is an action that minimizes posterior risk:

$$r(a^*) = \min_{a \in \mathcal{A}} r(a)$$

# Bayesian Point Estimation

- General Setup:
  - Data $\mathcal{D}$ generated by $p(y \mid \theta)$, for unknown $\theta \in \Theta$.

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# Important Cases

- Squared Loss : $\ell(\hat{\theta}, \theta) = \left(\theta - \hat{\theta}\right)^2 \quad \Rightarrow$ posterior mean
- Zero-one Loss: $\ell(\theta, \hat{\theta}) = 1(\theta \neq \hat{\theta}) \quad \Rightarrow$ posterior mode
- Absolute Loss : $\ell(\hat{\theta}, \theta) = \left|\theta - \hat{\theta}\right| \quad \Rightarrow$ posterior median (Exercise)

# Bayesian Point Estimation: Square Loss

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- **Bayes action** for **square loss** is the posterior mean.

# Bayesian Point Estimation: Zero-One Loss

- Suppose $\Theta$ is discrete (e.g. $\Theta = \{\text{english}, \text{french}\}$)
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- This $\hat{\theta}$ is called the **maximum a posteriori (MAP)** estimate.
- The MAP estimate is the **mode** of the posterior distribution.

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  - Only choices are
    - **family of distributions**, indexed by $\Theta$, and the
    - **prior distribution** on $\Theta$
  - For decision making, need a **loss function**.
  - Everything after that is **computation**.

# The Bayesian Method

1. **Define the model**:
   - Choose a parametric family of densities:
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