SVM Objective

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Slides based on Lecture Lab 3 from David Rosenberg's course material.

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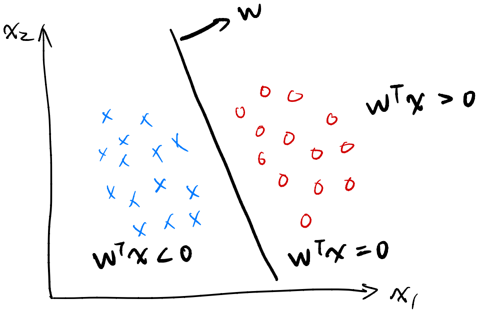
Maximum Margin Classifier

Linearly Separable Data

Consider a linearly separable dataset $\mathcal{D}$:

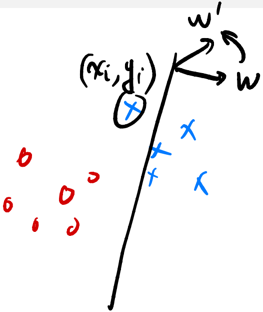


Find a separating hyperplane such that

- $w^T x_i > 0$ for all x_i where $y_i = +1$
- $w^T x_i < 0$ for all x_i where $y_i = -1$

The Perceptron Algorithm

- Initialize $w \leftarrow 0$
- While not converged (exists misclassified examples)
  - For $(x_i, y_i) \in \mathcal{D}$
    - If $y_i w^T x_i < 0$ (wrong prediction)
    - Update $w \leftarrow w + y_i x_i$

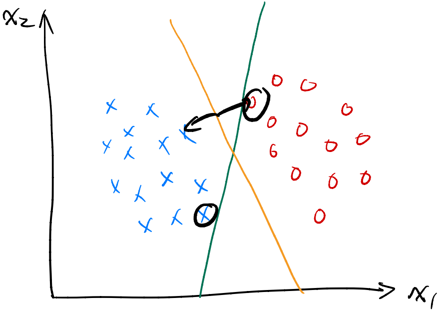


- Intuition: move towards misclassified positive examples and away from negative examples
- Guarantees to find a zero-error classifier (if one exists) in finite steps
- What is the loss function if we consider this as a SGD algorithm?

Maximum-Margin Separating Hyperplane

For separable data, there are infinitely many zero-error classifiers.

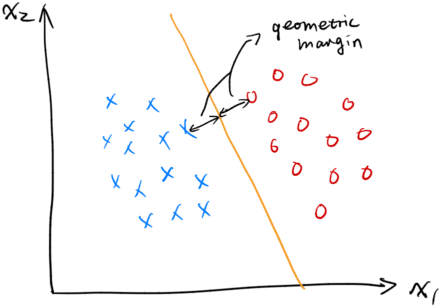
Which one do we pick?



(Perceptron does not return a unique solution.)

Maximum-Margin Separating Hyperplane

We prefer the classifier that is farthest from both classes of points



- Geometric margin: smallest distance between the hyperplane and the points
- Maximum margin: *largest* distance to the closest points

Geometric Margin

We want to maximize the distance between the separating hyperplane and the cloest points.

Let's formalize the problem.

Definition (separating hyperplane)

We say (x_i, y_i) for $i = 1, \ldots, n$ are **linearly separable** if there is a $w \in \mathbb{R}^d$ and $b \in \mathbb{R}$ such that $y_i(w^T x_i + b) > 0$ for all i. The set $\{v \in \mathbb{R}^d \mid w^T v + b = 0\}$ is called a **separating hyperplane**.

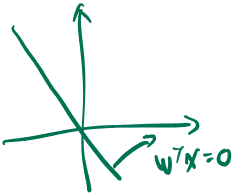
Definition (geometric margin)

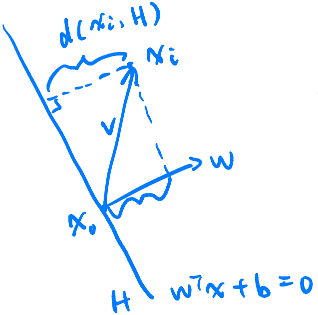
Let H be a hyperplane that separates the data (x_i, y_i) for $i = 1, \ldots, n$. The **geometric margin** of this hyperplane is

$$\min_i d(x_i, H),$$

the distance from the hyperplane to the closest data point.

Distance between a Point and a Hyperplane





$v = \alpha w + \beta w_\perp$

$$
\begin{aligned}
v &= x_i - x_0 \\
d(x_i, H) &= \mathrm{proj}(v, w) \\
&= \frac{v \cdot w}{\|w\|_2} \\
&= \frac{(x_i - x_0) \cdot w}{\|w\|_2} \\
&= \frac{x_i^T w + b}{\|w\|_2}
\end{aligned}
$$

- Projection of $v \in \mathbb{R}^d$ onto $w \in \mathbb{R}^d$: $\frac{v \cdot w}{\|w\|_2} = \alpha$
- Distance between x_i and H:

$$d(x_i, H) = \left| \frac{w^T x_i + b}{\|w\|_2} \right| = \frac{y_i(w^T x_i + b)}{\|w\|_2}$$

Maximize the Margin

We want to maximize the geometric margin:

$$\text{maximize} \underbrace{\min_i d(x_i, H)}.$$

Given separating hyperplane $H = \{v \mid w^T v + b = 0\}$, we have

$$\text{maximize} \underbrace{\min_i \frac{y_i(w^T x_i + b)}{\|w\|_2}}_{M}.$$

Let's remove the inner minimization problem by

$$\begin{array}{ll} \text{maximize} & M \\ \text{subject to} & \frac{y_i(w^T x_i + b)}{\|w\|_2} \geqslant M \quad \text{for all } i \end{array}$$

Note that the solution is not unique (why?). $w \leftarrow \alpha w \quad b \leftarrow \alpha b$

Maximize the Margin

Let's fix the norm $\|w\|_2$ to $1/M$ to obtain: $\|w\|_2 = \frac{1}{M}$

$$\begin{array}{ll} \text{maximize} & \frac{1}{\|w\|_2} \\ \text{subject to} & y_i(w^T x_i + b) \geqslant 1 \quad \text{for all } i \end{array}$$

It's equivalent to solving the minimization problem

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}\|w\|_2^2 \\ \text{subject to} & y_i(w^T x_i + b) \geqslant 1 \quad \text{for all } i \end{array}$$

Note that $y_i(w^T x_i + b)$ is the (functional) margin.

In words, it finds the minimum norm solution which has a margin of at least 1 on all examples.

Soft Margin SVM

What if the data is *not* linearly separable?

For any w, there will be points with a negative margin.

Introduce **slack variables** to penalize small margin:

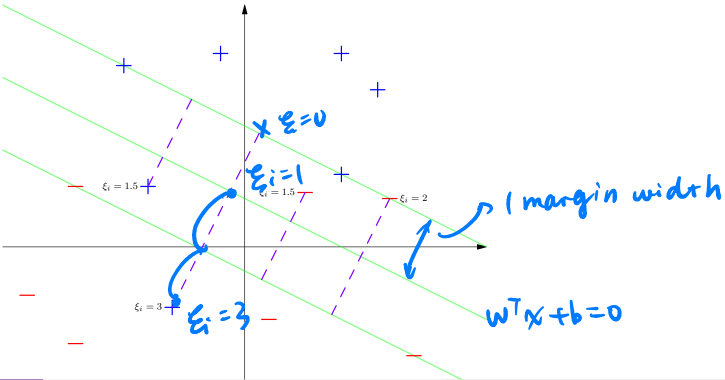
$$\begin{array}{ll} \text{minimize} & \frac{1}{2}\|w\|_2^2 + \frac{C}{n}\sum_{i=1}^n \xi_i \\ \text{subject to} & y_i(w^T x_i + b) \geqslant 1 - \xi_i \quad \text{for all } i \\ & \xi_i \geqslant 0 \quad \text{for all } i \end{array}$$

- If $\xi_i = 0 \, \forall i$, it's reduced to hard SVM.
- What does $\xi_i > 0$ mean?
- What does C control?

Slack Variables

$d(x_i, H) = \underbrace{\frac{y_i(w^T x_i + b)}{\|w\|_2}}_{M} \geqslant \frac{1-\xi_i}{\|w\|_2}$, thus ξ_i measures the violation by multiples of the geometric margin:

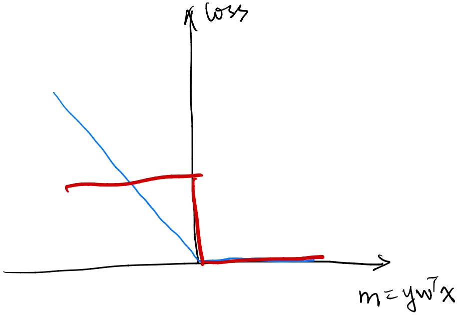
- $\xi_i = 1$: x_i lies on the hyperplane
- $\xi_i = 3$: x_i is past 2 margin width beyond the decision hyperplane



Minimize the Hinge Loss

Perceptron Loss

$$\ell(x, y, w) = \max(0, -yw^T x)$$



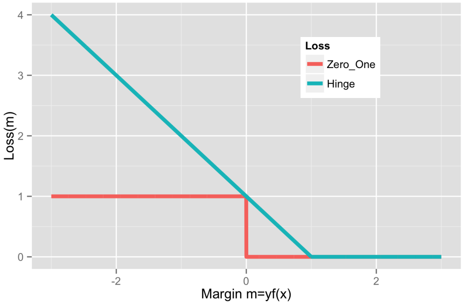
$w \leftarrow w + y_i x_i$

$w \leftarrow w - \eta \nabla f(w)$

If we do ERM with this loss function, what happens?

Hinge Loss

- SVM/Hinge loss: $\ell_{\text{Hinge}} = \max\{1-m, 0\} = (1-m)_+$
- Margin $m = yf(x)$; "Positive part" $(x)_+ = x\mathbb{1}(x \geqslant 0)$.



Hinge is a **convex**, **upper bound** on $0-1$ loss. Not differentiable at $m=1$.
We have a **"margin error"** when $m<1$.

Support Vector Machine

Using ERM:

- Hypothesis space $\mathcal{F} = \left\{ f(x) = w^T x + b \mid w \in \mathbf{R}^d, b \in \mathbf{R} \right\}$.
- ℓ_2 regularization (Tikhonov style)
- Hinge loss $\ell(m) = \max\{1 - m, 0\} = (1 - m)_+$ yw^Tx
- The SVM prediction function is the solution to

$$\min_{w \in \mathbf{R}^d, b \in \mathbf{R}} \underbrace{\overset{\lambda}{\frac{1}{2}}||w||^2}_{\text{regularizer}} + \underbrace{\overset{\frac{1}{n}}{\frac{c}{n}} \sum_{i=1}^{n} \max\left(0, 1 - y_i \left[w^T x_i + b\right]\right)}_{\text{ER}}.$$

- Not differentiable because of the max

SVM as a Constrained Optimization Problem

- The SVM optimization problem is equivalent to

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}||w||^2 + \frac{c}{n}\sum_{i=1}^{n}\xi_i \\ \text{subject to} & \xi_i \geqslant \max\left(0, 1 - y_i\left[w^T x_i + b\right]\right). \end{array}$$

- Which is equivalent to

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}||w||^2 + \frac{c}{n}\sum_{i=1}^{n}\xi_i \\ \text{subject to} & \xi_i \geqslant \left(1 - y_i\left[w^T x_i + b\right]\right) \text{ for } i = 1, \ldots, n \\ & \xi_i \geqslant 0 \text{ for } i = 1, \ldots, n \end{array}$$

Summary

Two ways to derive the SVM optimization problem:

- Maximize the (geometric) margin
- Minimize the hinge loss with ℓ_2 regularization

Both leads to the minimum norm solution satisfying certain margin constraints.

- **Hard-margin SVM**: all points must be correctly classified with the margin constraints
- **Soft-margin SVM**: allow for margin constraint violation with some penalty