Lagrangian Duality and Convex Optimization

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Optimization

General Optimization Problem: Standard Form

$x \in \mathbf{R}^n$ are the **optimization variables** and f_0 is the **objective function.**

$$\text{minimize} \quad f_0(x)$$

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- Can you think of examples?

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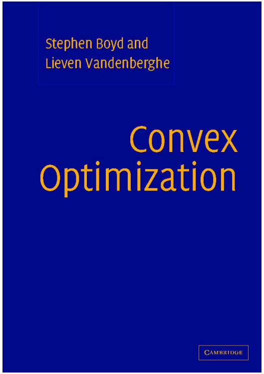
  - Main distinction is between **convex** and **non-convex** problems
  - Convex problems are the ones we know how to solve efficiently
  - Mostly batch methods until... around 2010? (earlier if you were into neural nets)
- By 2010 +- few years, most people understood the
  - optimization / estimation / approximation error tradeoffs
  - accepted that **stochatic methods** were often faster to get good results
    - (especially on big data sets)
  - now nobody's scared to try convex optimization machinery on non-convex problems

Your Reference for Convex Optimization

- Boyd and Vandenberghe (2004)
  - Very clearly written, but has a ton of detail for a first pass.
  - See the Extreme Abridgement of Boyd and Vandenberghe.



What we will quickly review today

1. Convex Sets and Functions
2. The General Optimization Problem
3. Lagrangian Duality: Convexity not required
4. Convex Optimization
5. Complementary Slackness

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Convex Sets and Functions

Notation from Boyd and Vandenberghe

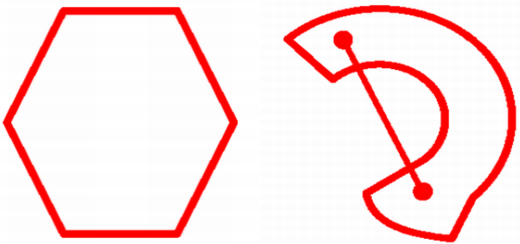
- $f : \mathbf{R}^p \to \mathbf{R}^q$ to mean that f maps from some *subset* of $\mathbf{R}^p$
    - namely $\mathbf{dom}\, f \subset \mathbf{R}^p$, where $\mathbf{dom}\, f$ is the domain of f

Convex Sets

Definition

A set C is **convex** if for any $x_1, x_2 \in C$ and any θ with $0 \leqslant \theta \leqslant 1$ we have

$$\theta x_1 + (1-\theta)x_2 \in C.$$



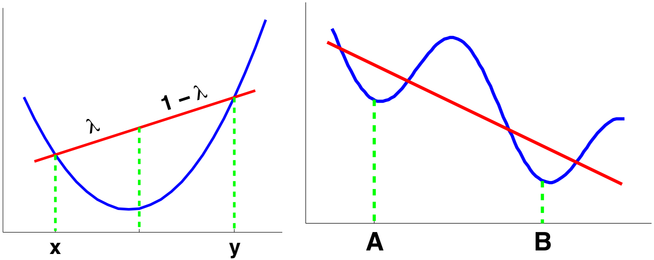
KPM Fig. 7.4

Convex and Concave Functions

Definition

A function $f : \mathbf{R}^n \to \mathbf{R}$ is **convex** if $\mathbf{dom}\ f$ is a convex set and if for all $x, y \in \mathbf{dom}\ f$, and $0 \leqslant \theta \leqslant 1$, we have

$$f(\theta x + (1-\theta)y) \leqslant \theta f(x) + (1-\theta) f(y).$$



KPM Fig. 7.5

Examples of Convex Functions on $\mathbf{R}$

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- Every norm on $\mathbf{R}^n$ is convex (e.g. $\|x\|_1$ and $\|x\|_2$)

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- $x \mapsto e^{ax}$ is convex on $\mathbf{R}$ for all $a \in \mathbf{R}$
- Every norm on $\mathbf{R}^n$ is convex (e.g. $\|x\|_1$ and $\|x\|_2$)
- Max: $(x_1, \ldots, x_n) \mapsto \max\{x_1 \ldots, x_n\}$ is convex on $\mathbf{R}^n$

Definition

A function f is **strictly convex** if the line segment connecting any two points on the graph of f lies **strictly** above the graph (excluding the endpoints).

Convex Functions and Optimization

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Consequences for optimization:

- **convex:** if there is a local minimum, then it is a **global** minimum
- **strictly convex:** if there is a local minimum, then it is the **unique global** minumum

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The General Optimization Problem

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where $x \in \mathbf{R}^n$ are the **optimization variables** and f_0 is the **objective function.**

Assume **domain** $\mathcal{D} = \bigcap_{i=0}^{m} \textbf{dom}\ f_i \cap \bigcap_{i=1}^{p} \textbf{dom}\ h_i$ is nonempty.

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- x^* is an **optimal point** (or a solution to the problem) if x^* is feasible and $f(x^*) = p^*$.

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- Consider an equality-constrained problem:

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- For simplicity, we'll drop equality contraints from this presentation.

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Lagrangian Duality: Convexity not required

The Lagrangian

The general [inequality-constrained] optimization problem is:

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leqslant 0, \;\; i = 1, \ldots, m \end{array}$$

Definition

The **Lagrangian** for this optimization problem is

$$L(x, \lambda) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x).$$

- λ_i's are called **Lagrange multipliers** (also called the **dual variables**).

The Lagrangian Encodes the Objective and Constraints

- Supremum over Lagrangian gives back encoding of objective and constraints:

$$\sup_{\lambda \succeq 0} L(x, \lambda) = \sup_{\lambda \succeq 0} \left(f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) \right)$$

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&= \begin{cases} f_0(x) & \text{when } f_i(x) \leqslant 0 \text{ all } i \\ \infty & \text{otherwise.} \end{cases}
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- Equivalent **primal form** of optimization problem:

$$p^* = \inf_x \sup_{\lambda \succeq 0} L(x,\lambda)$$

The Primal and the Dual

- Original optimization problem in **primal form:**

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- We will show **weak duality**: $p^* \geqslant d^*$ for any optimization problem

Weak Max-Min Inequality

Theorem

For ***any*** $f : W \times Z \to \mathbf{R}$, *we have*

$$\sup_{z \in Z} \inf_{w \in W} f(w,z) \leqslant \inf_{w \in W} \sup_{z \in Z} f(w,z).$$

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Proof

For any $w_0 \in W$ and $z_0 \in Z$, we clearly have

$$\inf_{w \in W} f(w,z_0) \leqslant f(w_0,z_0) \leqslant \sup_{z \in Z} f(w_0,z).$$

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Since $\inf_{w \in W} f(w,z_0) \leqslant \sup_{z \in Z} f(w_0,z)$ for all w_0 and z_0, we must also have

$$\sup_{z_0 \in Z} \inf_{w \in W} f(w,z_0) \leqslant \inf_{w_0 \in W} \sup_{z \in Z} f(w_0,z).$$

Weak Duality

- For any optimization problem (**not just convex**), weak max-min inequality implies **weak duality**:

$$
\begin{aligned}
p^* &= \inf_x \sup_{\lambda \succeq 0} \left[f_0(x) + \sum_{i=1}^m \lambda_i f_i(x) \right] \\
&\geqslant \sup_{\lambda \succeq 0, \nu} \inf_x \left[f_0(x) + \sum_{i=1}^m \lambda_i f_i(x) \right] = d^*
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- The difference $p^* - d^*$ is called the **duality gap**.
- For *convex* problems, we often have **strong duality**: $p^* = d^*$.

The Lagrange Dual Function

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$$d^* = \sup_{\lambda \succeq 0} \inf_x L(x, \lambda)$$

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- The dual function may take on the value $-\infty$ (e.g. $f_0(x) = x$).
- The dual function is always **concave**
  - since pointwise min of affine functions

The Lagrange Dual Problem: Search for Best Lower Bound

- In terms of Lagrange dual function, we can write weak duality as

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- So for any λ with $\lambda \geqslant 0$, **Lagrange dual function gives a lower bound on optimal solution**:

$$p^* \geqslant g(\lambda) \text{ for all } \lambda \geqslant 0$$

The Lagrange Dual Problem: Search for Best Lower Bound

- The **Lagrange dual problem** is a search for best lower bound on p^*:

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- Lagrange dual problem often easier to solve (simpler constraints).
- d^* can be used as stopping criterion for primal optimization.
- Dual can reveal hidden structure in the solution.

Recap Lagrangian Duality

- **Lagrangian** $L(x, \lambda) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x)$, with λ_i multipliers / dual variables

Recap Lagrangian Duality

- **Lagrangian** $L(x,\lambda) = f_0(x) + \sum_{i=1}^m \lambda_i f_i(x)$, with λ_i multipliers / dual variables
- Equivalence to original optimization problem:

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leqslant 0, \;\; i = 1, \ldots, m \end{array} \qquad \Rightarrow p^* = \inf_x \sup_{\lambda \succeq 0} \left[L(x,\lambda) \right]$$

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- **Dual function** $g(\lambda) = \inf_x L(x,\lambda) = \inf_x \left(f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x)\right)$ is always concave

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- **Weak duality** $p^* \geqslant \sup_{\lambda \succeq 0, \nu} \inf_x [L(x,\lambda)] = d^*$
- **Dual function** $g(\lambda) = \inf_x L(x,\lambda) = \inf_x \left(f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x)\right)$ is always concave
- Convex problems (f_i convex) have **strong duality** $p^* = d^*$

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Convex Optimization

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$$

where $f_0, \ldots, f_m$ are convex functions.

Strong Duality for Convex Problems

- For a convex optimization problems, we **usually** have strong duality, but not always.
    - For example:

$$
\begin{array}{ll}
\text{minimize} & e^{-x} \\
\text{subject to} & x^2/y \leqslant 0 \\
 & y > 0
\end{array}
$$

- The additional conditions needed are called **constraint qualifications**.

Example from Laurent El Ghaoui's EE 227A: Lecture 8 Notes, Feb 9, 2012

Slater's Constraint Qualifications for Strong Duality

- Sufficient conditions for strong duality in a **convex** problem.

[1] $\mathcal{D}$ is the set where all functions are defined, NOT the feasible set.

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Slater's Constraint Qualifications for Strong Duality

- Sufficient conditions for strong duality in a **convex** problem.
- Roughly: the problem must be **strictly** feasible.
- Qualifications when problem domain[1] $\mathcal{D} \subset \mathbf{R}^n$ is an open set:
  - **Strict feasibility is sufficient**. ($\exists x \; f_i(x) < 0$ for $i = 1, \ldots, m$)
  - For any affine inequality constraints, $f_i(x) \leqslant 0$ is sufficient.

[1] $\mathcal{D}$ is the set where all functions are defined, NOT the feasible set.

Slater's Constraint Qualifications for Strong Duality

- Sufficient conditions for strong duality in a **convex** problem.
- Roughly: the problem must be **strictly** feasible.
- Qualifications when problem domain[1] $\mathcal{D} \subset \mathbf{R}^n$ is an open set:
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- Otherwise, see notes or BV Section 5.2.3, p. 226.

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- Always have Lagrange multiplier is zero **or** constraint is active at optimum **or** both.

Complementary Slackness "Sandwich Proof"

- Assume strong duality: $p^* = d^*$ in a general optimization problem
- Let x^* be primal optimal and λ^* be dual optimal. Then:

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Each term in sum $\sum_{i=1} \lambda_i^* f_i(x^*)$ must actually be 0. That is

$$\boxed{\lambda_i^* f_i(x^*) = 0, \quad i = 1, \ldots, m}.$$

This condition is known as **complementary slackness**.

Result of "Sandwich Proof" and Consequences

- Let x^* be primal optimal and λ^* be dual optimal.
- If we have strong duality, then

$$p^* = d^* = f_0(x^*) = g(\lambda^*) = L(x^*, \lambda^*)$$

and we have **complementary slackness**

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- From the proof, we can also conclude that

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- If $x \mapsto L(x, \lambda^*)$ is differentiable, then we must have $\nabla L(x^*, \lambda^*) = 0$.